Available online at www.sciencedirect.com

SCIENCE DIRECT

ELSEVIER

Journal of APPLIED MATHEMATICS AND MECHANICS

www.elsevier.com/locate/jappmathmech

Journal of Applied Mathematics and Mechanics 68 (2004) 511–521

THE METHOD OF PROGRAMMED ITERATIONS IN ABSTRACT CONTROL PROBLEMS†

A. G. CHENTSOV

Ekaterinburg

e-mail: chentsov@imm.uran.ru

(*Received* 14 *January* 2004)

A direct version ("direct" in the sense of constructing control procedures – quasistrategies) of the method of programmed iterations is considered for the abstract problem of control of bundles of trajectories, information about which is realized by means of a certain signal. Conditions for the guaranteed solvability of the problem of encounter with a functional set, as defined by an iterative procedure in the space of multivalued responses to signals reaching the system, are investigated.

The procedure investigated here is known in control theory as the programmed iteration method (PIM). In essence, the PIM may also be considered in a broader context, as a method of constructing fixed points of operators which are extremal in the sense of some order relation. At the same time, historically speaking, the PIM is related to problems of the differential games theory [1–8], where it is used to construct value functions and stable bridges in N. N. Krasovskii's sense. Problems related to the use of programmed constructions to construct control strategies according to the feedback principle have traditionally occupied an important place in the research of the Krasovskii school. The ground of PIMs was prepared by the development of these programmed constructions of classical control theory, on the one hand, and by an essentially new theory of feedback control based on the fundamental Alternative Theorem of Krasovskii and Subbotin, on the other. This theory is associated with the use of very irregular (non-linear, discontinuous) control laws according to the feedback principle (Krasovskii's formalization), whose importance was convincingly demonstrated by N. N. Subbotina and A. I. Subbotin. It should be mentioned that in control problems with interference, earlier constructions of PIMs (see [9–12], etc.) were successfully combined with idealized control procedures – multivalued quasistrategies (see also [13, 14]); in connection with the concept of a "quasistrategy", we recall the papers [15, 16]. Also worthy of mention here are the studies [27–19] related to constructions of PIMs.

In control problems one frequently encounters the problem of incompleteness of information about phase states (in this connection see [3, 20–22]). General principles have been formulated for positional control in differential games with incomplete information [3]. Problems of programmed and positional control based on incomplete data, and minimax problems of observation and filtering, have been considered [20]. An important notion is that of an information set [20, p. 290], which has been widely used in control and observation problems.

In what follows an attempt is made to use the so-called direct version of the PIM in a control problem with incomplete information; we have in mind signal-based control and abstract analogues of signal-based control. As control procedures, we propose to use multivalued quasistrategies, that is, non-predictive responses to a signal; in the abstract version, we use the hereditariness of the response (an analogue of "non-predictiveness"). This particular construction of control procedures is related to an approach developed in [23–25] (the direct version of the PIM). The essence of these constructions, among which is the construction needed to construct a signal-based control quasistrategy, is to create an iterative procedure, each step of which essentially isolates the non-predictive part of an already constructed multifunction.

†*Prikl. Mat. Mekh.* Vol. 68, No. 4, pp. 573–585, 2004.

0021–8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2004.07.004

1. GENERAL CONCEPTS AND NOTATION

In what follows, for brevity, we shall use quantifiers and propositional connectives [26]. A family is a set all of whose elements are sets. The axiom of choice is assumed. The following general notation is essential for the direct version of the PIM in its abstract form.

We denote by $\mathcal{P}(H)$ ($\mathcal{P}'(H)$) the family of all (all non-empty) subsets of a set H, and by B^A the set [26] of all functions defined in A and range in B. If A and B are sets, $f \in B^A$ and $C \in \mathcal{P}(A)$, then $(f|C) \triangleq f \cap (C \times B) \in B^C$ is by defined the restriction of f to C; $(f|C)(x) = f(x)$ for $x \in C$. We denote by $\mathbb{R}$ the real line; $\mathcal{N} \triangleq \{1; 2; \ldots\}$ and $\mathcal{N}_0 \triangleq \{0\} \cup \mathcal{N} = \{0; 1; 2; \ldots\}$. With each operator defined in a given set we associate the sequence of its power: if A is a set and $T \in A^A$, then the sequence $(T^k)_{k \in \mathcal{N}_0}$ in the set A^A has the form

$$(T^0(a) \triangleq a \ \forall a \in A) \ \& \ (T^k = T \circ T^{k-1} \ \forall k \in \mathcal{N}) \tag{1.1}$$

In terms of (1.1), given $\alpha \in A$, we obtain a sequences $(T^k(\alpha))_{k \in \mathcal{N}_0}$ in A with the properties

$$(T^0(\alpha) = \alpha) \ \& \ (T^k(\alpha) = T(T^{k-1}(\alpha)) \ \forall k \in \mathcal{N}) \tag{1.2}$$

Iterative procedures of type (1.2) are used, in particular in schemes of the PIM. For any sets U and V, we set $\mathbb{M}(U, V) \triangleq \mathcal{P}(V)^U$; the elements of $\mathbb{M}(U, V)$ are also called multifunctions from U and V. One can, in particular, use the case $A = \mathbb{M}(U, V)$ in relations (1.1) and (1.2). In that case we introduce the infinite power of an operator defined in A.

Thus, if U and V are sets and T is an operator defined in $\mathbb{M}(U, V)$, then the operator T^∞, which is also defined in $\mathbb{M}(U, V)$, is defined by the rule: for $\mathcal{C} \in \mathbb{M}(U, V)$ and $u \in U$

$$T^\infty(\mathcal{C})(u) \triangleq \bigcap_{k \in \mathcal{N}_0} T^k(\mathcal{C})(u) \tag{1.3}$$

We let $\sqsubseteq$ denote the pointwise order relation defined in the space of multifunctions by embedding: if U and V are sets, and $\alpha \in \mathbb{M}(U, V)$ and $\beta \in \mathbb{M}(U, V)$, then $\alpha \sqsubseteq \beta$ stands for the statement

$$\alpha(u) \subset \beta(u) \ \forall u \in U$$

In these terms, we introduce the concept of isotonicity of operators defined in spaces of multifunctions. For any sets U and V, we let $\mathcal{M}[U; V]$ denote the set of all operators T defined in $\mathbb{M}(U, V)$ such that $T(\mathcal{C}) \sqsubseteq \mathcal{C} \ \forall \mathcal{C} \in \mathbb{M}(U, V)$. In what follows, we will need the concepts of monotone convergence of sets and multifunctions. The first concept is standard [27, Chap. I]: if U is a set, $(M_i)_{i \in \mathcal{N}}$ is a sequence of subsets of U (that is, a sequence in $\mathcal{P}(U)$) and $M \in \mathcal{P}(U)$, then $(M_i)_{i \in \mathcal{N}} \downarrow M$ means that (1) M is the intersection of all the sets M_i, $i \in \mathcal{N}$; (2) $M_{k+1} \subset M_k \ \forall k \in \mathcal{N}$. This type of convergence carries over to the case of multifunctions: if A and B are sets, $(\mathcal{C}_i)_{i \in \mathcal{N}}$ is a sequence in $\mathbb{M}(A, B)$ and $\mathcal{C} \in \mathbb{M}(A, B)$, then by definition

$$((\mathcal{C}_i)_{i \in \mathcal{N}} \Downarrow \mathcal{C}) \Leftrightarrow ((\mathcal{C}_i(a))_{i \in \mathcal{N}} \downarrow \mathcal{C}(a) \ \forall a \in A) \tag{1.4}$$

Thus, in spaces of type $\mathbb{M}(A, B)$ one uses the pointwise order relation $\sqsubseteq$ and pointwise convergence $\Downarrow$ (see (1.4)). We mention one simple corollary: if A and B are sets, $T \in \mathcal{M}[A; B]$ and $\mathcal{C} \in \mathbb{M}(A, B)$, then the sequence $(T^k(\mathcal{C}))_{k \in \mathcal{N}}$ (in $\mathbb{M}(A, B)$) converges to the multifunction $T^\infty(\mathcal{C})$ defined in (1.3), that is

$$(T^k(\mathcal{C})_{k \in \mathcal{N}} \Downarrow T^\infty(\mathcal{C}))$$

2. AN ABSTRACT CONTROL PROBLEM WITH INCOMPLETE INFORMATION

We shall consider a fairly general formulation of the problem of signal-based control, where the signal comes from a certain object. The control procedures are defined as multivalued quasistrategies, as applied in control problems with complete information (in this connection see [15, 16], where single-valued quasistrategies are used in problems of differential game theory). In connection with the following constructions, we note the example [28, pp. 357–360] of a certain class of signal-based control problems (we have in mind the game-theoretic problem of encounter up to a given time when the information about one of the objects is distorted by additive noise). The idea of the construction of the solution is then extended to the very general case of hereditary control procedures, but the hereditary property itself is no longer necessarily related to the traditional non-predictive property in the class of responses to a time function. This hereditariness property corresponds to a concept introduced in [23–25] (the traditional interpretation associated with the non-predictiveness of responses to the generated time functions will be described in Section 6).

Intuitively speaking, the problem considered below (in its general formulation) corresponds to a system developing in a certain abstract space; the control aspect of the system's operation has a well-defined goal, whose implementation is hindered by uncertainty as to the conditions in some part of the space. Concerning these conditions (which may be represented by a mapping defined in the abstract space) one has only indirect information, supplied as a signal produced by a mechanism which is not required to be hereditary in any sense. In the developing system under consideration, on the contrary, only hereditary procedures are admissible, corresponding in actual fact to the use of responses to the fragment of the signal received by the system. These fragments themselves depend on a previously specified family of non-empty sets (in the abstract space), whose points, generally speaking, do not have the sense of instants of time. Only the sets themselves, to which the interacting mappings are restricted, are essential. It is in terms of these restrictions that the property of hereditariness is defined, admitting of both interpretations that are traditional for the theory of dynamical systems and non-traditional interpretations (not formally associated with any dynamics). For example, one might speak of computing fragments of certain functions given indirect information about the analogous fragments of other functions, subsequently extending the domain of the space that determines those fragments.

In this very general situation, the actual operations performed may be interpreted as implementation of hereditary choice (in the aforementioned sense). The mechanisms for such choice (responses to a signal) may be considered as analogues of control procedures. However, uncontrollable factors (the characteristics of the space, formalized by mappings from some functional set, and also the possible versions of the signal itself) affect the actual implementation. One has to consider bundles of possible trajectories, understood of course in a broadened sense compared with the "usual" case of controlled dynamical systems. It is natural to require such a procedure of hereditary choice to furnish guarantees in the part relating to the achievement of the goal on trajectories of a bundle. We thus arrive at a game-theoretic formulation, as will be reflected in the discussion that follows.

From now on, X, Y, Υ and E will be arbitrary fixed non-empty sets. We use X as an analogue of the control interval. The sets Y, Υ and E will play the role of sets of values of functions defined on X. We fix

$$(Z \in \mathcal{P}'(Y^X)) \;\&\; (\mathbb{E} \in \mathcal{P}'(E^X))$$

where Z and $\mathbb{E}$ are two non-empty sets in function spaces: $Z \subset Y^X$, $\mathbb{E} \subset E^X$. Functions defined in Z and $\mathbb{E}$ play the part of trajectories; we assume that the choice of a "trajectory" $z \in Z$ is supervised by player I and the choice of a "trajectory" $e \in \mathbb{E}$ by player II. A set $M \in \mathcal{P}'(Z \times \mathbb{E})$, that is, a non-empty set $M, M \subset Z \times \mathbb{E}$, is given. Player I's goal is to bring about the event $(z, e) \in M$, but his information about the "trajectory" $e \in \mathbb{E}$ needed to choose $z \in Z$ is generally incomplete, that is: player I receives the signal $\omega \in \Upsilon^X$ (a function from X into Υ) generated by the "trajectory" e.

Thus, let us fix on operator

$$\Lambda : \mathbb{E} \to \mathcal{P}'(\Upsilon^X) \tag{2.1}$$

that is, a multifunction from $\mathbb{E}$ into Υ^X; if $e \in \mathbb{E}$, $\Lambda(e)$ is the set of all signals that are possible upon realization of e. Player I's actual choice of $\lambda \in \Lambda(e)$ is not predicted. Then

$$\Omega \triangleq \bigcup_{e \in \mathbb{E}} \Lambda(e) \in \mathcal{P}'(\Upsilon^X) \tag{2.2}$$

is the set of all possible signals (the choice of the "trajectory" $e \in \mathbb{E}$ is also not predicted for player I).

Thus, Ω is a non-empty set in the space $\Upsilon^X = \{X \to \Upsilon\}$ of all functions defined in X with range in Υ; information as to a function $\omega \in \Omega$ will be communicated to player I. Consequently, of course, he will also have information as to the non-empty set $\{e \in \mathbb{E} \mid \omega \in \Lambda(e)\}$.

We define an operator

$$\mathbb{L} : \Omega \to \mathcal{P}'(\mathbb{E}) \tag{2.3}$$

(that is, a multifunction from Ω into $\mathbb{E}$) by the logical rule: if $\omega \in \Omega$, then $\mathbb{L}(\omega) \triangleq \{e \in \mathbb{E} \,|\, \omega \in \Lambda(e)\}$. Definition (2.3) is actually a rule for constructing functional information sets. The operator (2.3) may be considered, in a sense, as the inverse of Λ (2.1).

We define $\alpha^0 \in \mathbb{M}(\Omega, Z)$ by the rule

$$\alpha^0(\omega) \triangleq \{z \in Z \mid (z, e) \in M \ \forall e \in \mathbb{L}(\omega)\}, \quad \forall \omega \in \Omega \tag{2.4}$$

We now have an object multifunction from Ω into Z. The meaning of the definition (2.4) of α^0 is the following. If a signal ω has been made available to player I, he is forced to admit of the possibility that any "trajectory" $e \in \mathbb{E}$ capable of producing that signal may have been chosen. Player I's goal will be achieved with a guarantee, if the required encounter with the set M is assured for all "trajectories" in $\mathbb{L}(\omega)$, since he must prepare for the possibility that player II will have taken the most unsatisfactory actions. Moreover, the signal ω itself may be produced in a way that is unsatisfactory for player I. Finally, in typical specific settings of this kind, the trajectory $z \in Z$ must be produced "in real time", as the signal fragments arrive. In the very general formulation considered here, we introduce an abstract analogue of this requirement.

Let $\mathcal{X}$ be a given non-empty family of non-empty subsets of X: $\mathcal{X} \in \mathcal{P}'(\mathcal{P}'(X))$. A special notion of hereditariness is defined in terms of $\mathcal{X}$, namely, $\mathcal{X}$-hereditariness: if $\alpha \in \mathbb{M}(\Omega, Z)$, we call α hereditary multifunction (from Ω to Z) if $\forall \omega_1 \in \Omega$, $\forall \omega_2 \in \Omega$, $\forall A \in \mathcal{X}$

$$((\omega_1|A) = (\omega_2|A)) \Rightarrow (\{(z|A) : z \in \alpha(\omega_1)\} = \{(z|A) : z \in \alpha(\omega_2)\}) \tag{2.5}$$

Remark. In the case $X = [t_0, \vartheta_0]$, where $t_0 \in \mathbb{R}$, $\vartheta_0 \in \,]t_0, \infty[$ and $\mathcal{X} = \{[t_0, t] : t \in X\}$, formula (2.5) defines the traditional condition for the response α to be unpredictable (in multi-valued form).

In connection with condition (2.5) we mention the general constructions of [23–25] and [28, Chap. 6]. We recall some of them. If $A \in \mathcal{X}$, then

$$(\Omega_0(\omega|A) \triangleq \{\nu \in \Omega \,|\, (\omega|A) = (\nu|A)\} \ \forall \omega \in \Omega) \& (Z_0(z|A) \triangleq \{z' \in Z \,|\, (z|A) = (z'|A)\} \ \forall z \in Z)$$

and in these terms we define $\mathbf{\Gamma}_A \in \mathcal{M}[\Omega, Z]$ as the mapping defined in $\mathbb{M}(\Omega, Z)$ by the rule $\forall \zeta \in \mathbb{M}(\Omega, Z)$, $\forall \omega \in \Omega$

$$\mathbf{\Gamma}_A(\zeta)(\omega) \triangleq \{z \in \zeta(\omega) \,|\, Z_0(z|A) \cap \zeta(\nu) \neq \varnothing \ \forall \nu \in \Omega_0(\omega|A)\} \tag{2.6}$$

With the operators $\mathbf{\Gamma}_A, A \in \mathcal{X}$, we associate in the natural way (see [23–25]) the operator $\mathbf{\Gamma} \in \mathcal{M}[\Omega, Z]$ such that

$$\mathbf{\Gamma}(\zeta)(\omega) \triangleq \bigcap_{A \in \mathcal{X}} \mathbf{\Gamma}_A(\zeta)(\omega) \ \forall \zeta \in \mathbb{M}(\Omega, Z), \ \forall \omega \in \Omega \tag{2.7}$$

Definitions (2.6) and (2.7) correspond to those introduced in [23–25].

The operator $\mathbf{\Gamma}$ plays an important part in relation to the hereditariness condition: a multifunction $\alpha \in \mathbb{M}(\Omega, Z)$ is hereditary in the sense of (2.5) if any only if $\alpha = \mathbf{\Gamma}(\alpha)$.

Thus, the hereditary multifunctions are precisely the fixed points of $\mathbf{\Gamma}$. Thus, $\mathbb{N} \triangleq \{\alpha \in \mathbb{M}(\Omega, Z) \,|\, \alpha = \mathbf{\Gamma}(\alpha)\}$ is the set of all hereditary multifunctions from Ω to Z. Note that (see [24, 25])

$$\mathbf{\Gamma}_A \circ \mathbf{\Gamma}_A = \mathbf{\Gamma}_A \ \forall A \in \mathcal{X} \tag{2.8}$$

that is, the operator (2.6) has the useful property of idempotency. As a corollary

$$\{\alpha \in \mathbb{M}(\Omega, Z) \,|\, \alpha = \mathbf{\Gamma}_A(\alpha)\} = \{\mathbf{\Gamma}_A(\beta) : \beta \in \mathbb{M}(\Omega, Z)\} \ \forall A \in \mathcal{X} \tag{2.9}$$

For further constructions it makes sense to associate relations (2.8) and (2.9) with the multifunction α^0 of (2.4). Define

$$\mathbb{N}^0 \triangleq \{\alpha \in \mathbb{N} \,|\, \alpha \sqsubseteq \alpha^0\} \tag{2.10}$$

The elements of set (2.10) are precisely the hereditary multiselectors α^0. Set (2.10) has a $\sqsubseteq$-greatest element $(\mathrm{na})[\alpha^0] \in \mathbb{N}^0$; with this notation

$$(\mathrm{na})[\alpha^0](\omega) \triangleq \bigcup_{\mathscr{C} \in \mathbb{N}^0} \mathscr{C}(\omega) \ \forall \omega \in \Omega \tag{2.11}$$

Note that (see (2.8) and (2.9) and the arguments of [24, 25]) set (2.10) admits of a representation in terms of the action of the operators $\mathbf{\Gamma}_A, A \in \mathscr{X}$, namely

$$\mathbb{N}^0 \triangleq \bigcap_{A \in \mathscr{X}} \{\mathbf{\Gamma}_A(\mathscr{C}) \colon \mathscr{C} \in \mathbb{M}(\Omega, Z), \mathscr{C} \sqsubseteq \alpha^0\} \tag{2.12}$$

and the multifunction $(\mathrm{na})[\alpha^0] \in \mathbb{N}^0$ is such that $\mathscr{X} \sqsubseteq (\mathrm{na})[\alpha^0] \ \forall \mathscr{X} \in \mathbb{N}^0$. Thus (see (2.12)), to define multifunction (2.11), one has to construct the set

$$\mathbf{S}^0 \triangleq \{\mathscr{C} \in \mathbb{M}(\Omega, Z) | \mathscr{C} \sqsubseteq \alpha^0\} \tag{2.13}$$

and then consider the images of set (2.13) under each of the operators $\mathbf{\Gamma}_A, A \in \mathscr{X}$. The intersection of these images (which coincides with $\mathbb{N}^0$) contains the multifunction $(\mathrm{na})[\alpha^0]$ of (2.11). Moreover, this multifunctions is the $\sqsubseteq$-greatest element of the intersection. This approach thus reduces to inspecting the multifunctions in each of the aforementioned images of the set (2.13) and choosing the $\sqsubseteq$-greatest of them.

If $\alpha \in \mathbb{M}(\Omega, Z)$, we define

$$(\mathrm{DOM})[\alpha] \triangleq \{\omega \in \Omega | \alpha(\omega) \neq \varnothing\} \tag{2.14}$$

(the effective domain of the multifunction α). In terms of $\mathbb{N}$ and (2.14), we construct the set of (multi-valued) quasistrategies: let Q denote the set of all $\alpha \in \mathbb{N}$ such that $(\mathrm{DOM})[\alpha] = \Omega$. As the basic question we consider that of guaranteed encounter, realizable by a suitable quasistrategy, with the set M. In this connection we give special consideration to the question of solvability and, when that question has a positive answer, the problem of constructing a suitable quasistrategy.

In this formulation, then, the quasistrategies are precisely the non-degenerate hereditary multifunctions. Hence, by definition (2.4)

$$Q^0 \triangleq \{\alpha \in \mathbb{N}^0 | (\mathrm{DOM})[\alpha] = \Omega\} \tag{2.15}$$

is the set of all quasistrategies for which the "trajectories" will surely encounter M for any realization of the signal. In rigorous terms, the following proposition holds for Q^0 (2.15).

Proposition 1. Q^0 is the set of all quasistrategies $\alpha \in Q$ for each of which

$$(z, e) \in M \ \forall e \in \mathbb{E}, \quad \forall \omega \in \Lambda(e), \quad \forall z \in \alpha(\omega)$$

We confine ourselves to an outline of the proof, setting

$$Q_0 \triangleq \{\alpha \in Q | (z, e) \in M \ \forall e \in \mathbb{E}, \forall \omega \in \Lambda(e), \forall z \in \alpha(\omega)\}$$

Choose an arbitrary $\beta^0 \in Q^0$, which means that (for a multifunction $\beta^0 \in Q$) the following property holds

$$\beta^0(\omega) \subset \alpha^0(\omega) \ \forall \omega \in \Omega$$

Let $e_* \in \mathbb{E}$; then $\Lambda(e_*) \in \mathscr{P}'(\Omega)$ by definition (2.2), that is, $\Lambda(e_*)$ is a non-empty subset of Ω. Choose $\omega_* \in \Lambda(e_*)$. Then $\beta^0(\omega_*) \subset \alpha^0(\omega_*)$. In addition, $e_* \in \mathbb{L}(\omega_*)$ by the definition of the operator (2.3). Taking definition (2.4) into account, we now obtain

$$(z, e_*) \in M \ \forall z \in \beta^0(\omega_*)$$

Since the element ω_* was chosen arbitrarily, we have

$$(z, e_*) \in M \ \forall \omega \in \Lambda(e_*), \forall z \in \beta^0(\omega)$$

But the choice of the element e_* was also arbitrary. Thus $\beta^0 \in Q_0$, since β^0 is a quasistrategy. This establishes the embedding $Q^0 \subset Q_0$.

Let $\beta_0 \in Q_0$. Then, in particular, $\beta_0 \in Q$. Choose an arbitrary $\omega^* \in \Omega$. Then

$$\mathbb{L}(\omega^*) = \{e \in \mathbb{E} | \omega^* \in \Lambda(e)\}, \quad \mathbb{L}(\omega') \neq \varnothing$$

Let $e^* \in \mathbb{L}(\omega^*)$. Thus, $e^* \in \mathbb{E}$ and at the same time $\omega^* \in \Lambda(e^*)$.

Thus, $e^* \in \mathbb{E}$ and $\omega^* \in \Lambda(e^*)$, which by the definition of Q_0 implies

$$(z, e^*) \in M \ \forall z \in \beta_0(\omega^*)$$

Since the element e^* was chosen arbitrarily, we have shown that

$$(z, e) \in M \ \forall e \in \mathbb{L}(\omega^*), \quad \forall z \in \beta_0(\omega^*)$$

As a corollary, definition (2.4) implies the embedding $\beta_0(\omega^*) \subset \alpha^0(\omega^*)$. Since the choice of the element ω^* was arbitrary, we obtain the property $\beta_0 \sqsubseteq \alpha^0$ and, as a corollary, $\beta_0 \in \mathbb{N}^0$ (see (2.10)). But β_0 is a quasistrategy, and therefore $(\mathrm{DOM})[\beta_0] = \Omega$ and, by definition (2.15), $\beta_0 \in Q^0$. The embedding $Q_0 \subset Q^0$ has thus been established, which is what was required.

By property (2.15)

$$((\mathrm{na})[\alpha^0] \in Q^0) \Leftrightarrow ((\mathrm{DOM})[(\mathrm{na})[\alpha^0]] = \Omega) \tag{2.16}$$

Taking the extremality property of $(\mathrm{na})[\alpha^0]$ into consideration (see definition (2.11)), we infer from definition (2.15) that

$$((\mathrm{na})[\alpha^0] \notin Q) \Leftrightarrow (Q^0 = \varnothing) \tag{2.17}$$

Properties (2.16) and (2.17) determine the role of $(\mathrm{na})[\alpha^0]$ in problems relating to the solvability of the basic problem in the class of quasistrategies.

Proposition 2. If $(\mathrm{DOM})[(\mathrm{na})[\alpha^0]] \neq \Omega$, then $\forall \alpha \in Q \ \exists e \in \mathbb{E} \ \exists \omega \in \Lambda(e) \ \exists z \in \alpha(\omega)$: $(z, e) \notin M$.

The proof follows easily from relation (2.17).

Properties (2.16) and (2.17) and Propositions 1 and 2 imply the following.

Theorem 1. The following properties are equivalent: (1) $(\mathrm{na})[\alpha^0] \in Q^0$, (2) $Q^0 \neq \varnothing$.

Thus, the basic problem is successfully solvable in the class of quasistrategies if and only if (see property (2.16)) and the sets (2.11) are non-empty. At the same time, representation (2.12) also implies the following property: if a multifunction $\mathfrak{D}^0 \in \mathbb{M}(\Omega, Z)$ exists for which $(\mathrm{DOM})[\mathfrak{D}^0] = \Omega$, and in addition the condition of membership in the image of the set (2.13) under the action of every operator $\mathbf{\Gamma}_A$, $A \in \mathscr{X}$ is satisfied, then $Q^0 \neq \varnothing$; when that happens, $\mathfrak{D}^0 \in Q^0$. In connection with analogues of this approach associated with the transformation S^0 of (2.13), we mention certain relations obtained in [29, 30].

3. THE METHOD OF ITERATIONS

Representation (2.12) indicates an approach to the definition of a multifunction in $\mathbb{N}^0$ and, in particular, to the definition of $(\mathrm{na})[\alpha^0]$; the latter multifunction contains exhaustive information about the possibility of solving the basic problem in the class of quasistrategies (see Theorem 1). As a whole, representation (2.12) has the sense of experimentation with responses to possible signals subject to the multifunction α^0; the main role here is that of actions based on operators (2.6). A more regular method is obtained by having recourse to definition (2.7) and carrying out an iterative procedure on that basis.

Consider the sequence $(\mathbf{\Gamma}^k)_{k \in \mathcal{N}_0}$ of powers of $\mathbf{\Gamma}$ and the limiting operator $\mathbf{\Gamma}^\infty$ (see (1.3)). Then $(\mathbf{\Gamma}^k(\alpha^0))_{k \in \mathcal{N}_0}$ is a sequence in $\mathbb{M}(\Omega, Z)$ for which

$$(\mathbf{\Gamma}^0(\alpha^0) = \alpha^0) \& (\mathbf{\Gamma}^k(\alpha^0) = \mathbf{\Gamma}(\mathbf{\Gamma}^{k-1}(\alpha^0)) \ \forall k \in \mathcal{N}) \tag{3.1}$$

The sequence (3.1) converges (pointwise) to a multifunction $\mathbf{\Gamma}^\infty(\alpha^0) \in \mathbb{M}(\Omega, Z)$

$$(\mathbf{\Gamma}^k(\alpha^0))_{k \in \mathcal{N}} \Downarrow \mathbf{\Gamma}^\infty(\alpha^0) \tag{3.2}$$

where $(\mathrm{na})[\alpha^0] \sqsubseteq \mathbf{\Gamma}^\infty(\alpha^0)$ (see [23–25]). Consideration of property (2.16) and Theorem 1 yields the following.

Proposition 3. If $(\mathrm{DOM})[\mathbf{\Gamma}^\infty(\alpha^0)] \neq \Omega$, then $Q^0 \neq \varnothing$.

Thus, an iterative method based on relations (3.1) and (3.2) (the direct version of the PIM) may always be used to obtain sufficient conditions for the basic problem to be solvable in the class of quasistrategies. In fact, in many cases the convergence in (3.2) actually yields $(\mathrm{na})[\alpha^0]$. Conditions of this kind were described in [23–25]. For the moment we will recall only a few general arguments.

Let $\mathfrak{Z}$ denote the family of all sets $\mathbf{S}$, $\mathbf{S} \subset \mathbb{M}(\Omega, Z)$, such that $\mathbf{\Gamma}(\mathscr{C}) \in \mathbf{S}\ \forall \mathscr{C} \in \mathbf{S}$ (this is the definition of the family of all $\mathbf{\Gamma}$-invariant subsets of $\mathbb{M}(\Omega, Z)$). Let $\mathbf{Z}$ denote the family of all sets $\mathfrak{H}$, $\mathfrak{H} \subset \mathbb{M}(\Omega, Z)$, such that $\forall (H_i)_{i \in \mathcal{N}} \in \mathfrak{H}^{\mathcal{N}}\ \forall H \in \mathbb{M}(\Omega, Z)$

$$((H_i)_{i \in \mathcal{N}} \Downarrow H) \Rightarrow (H \in \mathfrak{H})$$

$\mathbf{Z}$ may be regarded as the family of all subsets of $\mathbb{M}(\Omega, Z)$ that are (sequentially) closed under $\Downarrow$-convergence. Finally, let $\mathfrak{G}$ denote the family of all sets $\mathbb{U}$, $\mathbb{U} \subset \mathbb{M}(\Omega, Z)$, such that $\forall (U_j)_{j \in \mathcal{N}} \in \mathbb{U}^{\mathcal{N}}\ \forall U \in \mathbb{M}(\Omega, Z)$

$$((U_j)_{j \in \mathcal{N}} \Downarrow U) \Rightarrow ((\mathbf{\Gamma}(U_j))_{j \in \mathcal{N}} \Downarrow \mathbf{\Gamma}(U))$$

In the latter case we are considering subsets of $\mathbb{M}(\Omega, Z)$ on which the operator $\mathbf{\Gamma}$ has a certain analogue of the property of sequential continuity. Note that if $\mathbf{U} \in \mathfrak{Z}$ and $\alpha^0 \in \mathbf{U}$, then $(\mathbf{\Gamma}^k(\alpha^0))_{k \in \mathcal{N}_0}$ is a sequence in $\mathbf{U}$; if in addition $\mathbf{U} \in \mathbf{Z}$, then $\mathbf{\Gamma}^\infty(\alpha^0) \in \mathbf{U}$. It is known [24, 25] that if $\mathbf{S} \in \mathfrak{Z} \cap \mathfrak{G}$ and $\alpha^0 \in \mathbf{S}$, then

$$\mathbf{\Gamma}^\infty(\alpha^0) = (\mathrm{na})[\alpha^0] \tag{3.3}$$

but if in addition $\mathbf{S} \in \mathbf{Z}$, then $(\mathrm{na})[\alpha^0] \in \mathbf{S}$.

Consideration of (3.3) yields the following.

Theorem 2. Let $\exists \mathbf{S} \in \mathfrak{Z} \cap \mathfrak{G}$: $\alpha^0 \in \mathbf{S}$. Then the following three statements are equivalent: (1) $(\mathrm{DOM})[\mathbf{\Gamma}^\infty(\alpha^0)] = \Omega$, (2) $\mathbf{\Gamma}^\infty(\alpha^0) \in Q^0$, (3) $Q^0 \neq \varnothing$.

Thus, to determine whether it is possible to solve the basic problem successfully in the class of quasistrategies, one must verify the properties of the multifunction (2.4), carry out the iterative process (3.1), (3.2), and determine whether the effective domain of its limiting multifunction coincides with Ω.

In the next section it will be shown that the conditions imposed on the multifunction α^0 in Theorem 2 are frequency quite easy to verify.

4. TOPOLOGICAL CONSTRUCTIONS AND THE METHOD OF ITERATIONS

Let us consider the common case in which player I's "trajectory" is realized in a Hausdorff topological space. Thus, let the set Y (see Section 2) be endowed with a topology τ such that (Y, τ) is a Hausdorff space [31, p. 98].† Let $\otimes^X(\tau)$ be the topology of the set Y^X corresponding to the Tikhonov product of copies of the space (Y, τ) with index set X (see [31, p. 127]). Let ϑ denote the topology of the set Z induced [31, p. 77] from

$$(Y^X, \otimes^X(\tau)) \tag{4.1}$$

Consequently, (Z, ϑ) is a subspace of the topological space (4.1) and ϑ is the pointwise convergence topology in Z [31, p. 283]. Let $\mathbb{K}$ (resp. $\mathscr{K}$) denote the family of all compact [32, p. 196] (resp. sequentially compact, [32, p. 314]) subsets of Z in the topological space (Z, ϑ). This topological space is Hausdorff, and each set in $\mathbb{K}$ is compact [32, p. 208] in the topology induced from (Z, ϑ). Recall that $\mathbb{K}^\Omega$ and $\mathscr{K}^\Omega$ are the sets of all mappings from Ω into $\mathbb{K}$ and $\mathscr{K}$, respectively. It follows from general arguments [23–25] that

$$(\mathbb{K}^\Omega \in \mathfrak{Z} \cap \mathbf{Z} \cap \mathfrak{G}) \& (\mathscr{K}^\Omega \in \mathfrak{Z} \cap \mathbf{Z} \cap \mathfrak{G}) \tag{4.2}$$

†Editor's note. For [31] and [32], the page numbers refer to the Russian editions of these books.

Property (4.2) may now be used in Theorem 2. First, however, we point out a simple corollary of properties (3.3) and (4.2).

Proposition 4. If $\alpha^0 \in \mathbb{K}^\Omega$, then $\mathbf{\Gamma}^\infty(\alpha^0) = (\mathrm{na})[\alpha^0] \in \mathbb{K}^\Omega$. In the case $\alpha^0 \in \mathcal{K}^\Omega$, we have $\mathbf{\Gamma}^\infty(\alpha^0) = (\mathrm{na})[\alpha^0] \in \mathcal{K}^\Omega$.

As a supplement of Proposition 4 we mention a theorem [28, Theorem 6.1.2] on the basis of which one can achieve a combination of fragments α^0 with compact and sequentially compact values. Theorem 2 and property (4.2) imply the following theorem.

Theorem 3. If $(\alpha^0 \in \mathbb{K}^\Omega) \vee (\alpha^0 \in \mathcal{K}^\Omega)$, then statements 1–3 of Theorem 2 are equivalent.

In connection with the conditions imposed on α^0, there is also a useful addition to representations (2.12) and (2.13). By analogy with property (4.2), it can be shown [28, p. 355] that, if $A \in \mathcal{X}$, one has the following invariance properties: (1) $\{\mathbf{\Gamma}_A(\mathcal{C})\colon \mathcal{C} \in \mathbb{K}^\Omega\} \subset \mathbb{K}^\Omega$; (2) $\{\mathbf{\Gamma}_A(\mathcal{D})\colon \mathcal{D} \in \mathcal{K}^\Omega\} \subset \mathcal{K}^\Omega$. As a corollary, it follows from (2.12) that

$$\mathbb{N}^0 \cap \mathbb{K}^\Omega = \bigcap_{A \in \mathcal{X}} \{\mathbf{\Gamma}_A(\mathcal{H})\colon \mathcal{H} \in \mathbb{K}^\Omega,\ \mathcal{H} \sqsubseteq \alpha^0\} \tag{4.3}$$

$$\mathbb{N}^0 \cap \mathcal{K}^\Omega = \bigcap_{A \in \mathcal{X}} \{\mathbf{\Gamma}_A(\mathcal{H})\colon \mathcal{H} \in \mathcal{K}^\Omega,\ \mathcal{H} \sqsubseteq \alpha^0\} \tag{4.4}$$

Using a priori information about α^0, one can now use relations (4.3) and (4.4) to implement an analogue of the constructions relating to representation (2.13).

In fact, if $\alpha^0 \in \mathbb{K}^\Omega$ (that is, α^0 is a compact-valued multifunction), one has to define

$$\mathbf{S}^0_{\mathbb{K}} \triangleq \{\alpha \in \mathbb{K}^\Omega \mid \alpha \sqsubseteq \alpha^0\} \tag{4.5}$$

and construct the images of this set under each of the operators $\mathbf{\Gamma}_A, A \in \mathcal{X}$. The intersection of all these images contains $(\mathrm{na})[\alpha^0]$ and, in addition, $(\mathrm{na})[\alpha^0]$ is the $\sqsubseteq$-greatest element of the intersection. In general, if one can find in the aforementioned intersection of images a multifunction $\mathcal{D}^0_{\mathbb{K}} \in \mathbb{K}^\Omega$ with the property $(\mathrm{DOM})[\mathcal{D}^0_{\mathbb{K}}] = \Omega$, then $Q^0 \neq \varnothing$ (provided that α^0 is compact-valued).

But if $\alpha^0 \in \mathcal{K}^\Omega$, then one uses relation (4.4) to the same end: define

$$\mathbf{S}^0_{\mathcal{K}} \triangleq \{\alpha \in \mathcal{K}^\Omega \mid \alpha \sqsubseteq \alpha^0\} \tag{4.6}$$

and construct the images of this set under each of the operators $\mathbf{\Gamma}_A, A \in \mathcal{X}$. Then $(\mathrm{na})[\alpha^0]$ is the $\sqsubseteq$-greatest element of the intersection of all these images. If the intersection contains a multifunction $\mathcal{D}^0_{\mathcal{K}} \in \mathcal{K}^\Omega$ for which $(\mathrm{DOM})[\mathcal{D}^0_{\mathcal{K}}] = \Omega$, then $Q^0 \neq \varnothing$.

We have thus established criteria for the successful solvability of the basic problem without using the method of iterations.

5. THE CONDITIONS FOR THE BASIC PROBLEM TO BE UNSOLVABLE IN THE CLASS OF QUASISTRATEGIES

A more general sufficient condition for the problem in Section 2 to be unsolvable is contained in Proposition 2. Necessary and sufficient conditions in the appropriate classes of multifunctions α^0 may be derived from Theorems 2 and 3. They all relate to the problem of observing the situation $\mathbf{\Gamma}^\infty(\alpha^0)(\omega) = \varnothing$ for a certain realization $\omega \in \Omega$. We will now consider a few further possibilities in this direction, appealing to general arguments (see [23–25] and [25, Chap. 6]) relating to the factorization of Ω.

Let $\mathbb{H}_0$ be defined as the family of all sets $H \in \mathcal{P}'(\Omega)$ such that

$$\bigcup_{A \in \mathcal{X}} \Omega_0(\omega \mid A) \subset H \quad \forall \omega \in H \tag{5.1}$$

Now let $\mathbf{H}_0$ be the set of all families $\mathcal{U} \in \mathcal{P}'(\mathbb{H}_0)$ such that: (1) Ω is the union of all sets $U \in \mathcal{U}$, (2) $(U_1 \cap U_2 \neq \varnothing) \Rightarrow (U_1 = U_2)\ \forall U_1 \in \mathcal{U}, \forall U_2 \in \mathcal{U}$. The elements of $\mathbf{H}_0$ are partitions of the signal space into a sum of non-empty sets with property (5.1). In terms of $\mathbf{H}_0$ one can implement (see, for example, [28, Sec. 6.8]) a system of local iterative processes which are then "glued together" into a procedure (3.1), (3.2). These processes correspond to cells of the corresponding partitions in $\mathbf{H}_0$. This useful possibility will not be considered now. We shall consider only how to use these same partitions and the factorizations associated with them to look for ways to facilitate the verification of the property $Q^0 = \varnothing$. In this connection we note that, if $(\alpha \in \mathbb{K}^{\Omega}) \vee (\alpha \in \mathcal{K}^{\Omega})$, the following property holds for $\omega \in \Omega$: if $(\text{na})[\alpha^0](\omega) = \varnothing$, then $\exists n \in \mathcal{N}$: $\mathbf{\Gamma}^n(\alpha^0)(\omega) = \varnothing$. In fact, in this proposition one can also use certain conditions imposed on α^0 that admit of a combination of fragments of this multifunction with compact and sequentially compact values [28, p. 347].

We henceforth assume that the following condition holds.

Condition 1. The family $\mathcal{X}$ is the basis of a filter in X, that is

$$\forall A \in \mathcal{X}\ \forall B \in \mathcal{X}\ \exists C \in \mathcal{X}: C \subset A \cap B$$

Remark. In control problems, a typical situation is $X = [t_0, \vartheta_0]$, where $t_0 < \vartheta_0$ are two numbers and $\mathcal{X}$ is the family of all closed intervals $[t_0, t]$, $t \in X$. Condition 1 is satisfied in that case.

It is evident that in the case under consideration

$$\mathfrak{S} \triangleq \left\{ \bigcup_{A \in \mathcal{X}} \Omega_0(\omega | A) : \omega \in \Omega \right\} \in \mathbf{H}_0 \tag{5.2}$$

This defines a factorization of Ω; in fact, the factorization is extremal in $\mathbf{H}_0$: the partition $\mathfrak{S}$ is inscribed [32, pp. 195, 196] in any set $\mathcal{U} \in \mathbf{H}_0$. If $\alpha \in \mathbb{M}(\Omega, Z)$ and $S \in \mathfrak{S}$, then

$$((\text{DOM})[\mathbf{\Gamma}(\alpha)] \cap S \neq \varnothing) \Rightarrow (S \subset (\text{DOM}[\alpha])) \tag{5.3}$$

Property (5.3) of the partition (5.2) (it has been verified – see [24] and [28, Sec. 6.9]) makes it possible to provide a few prescriptions relating to the question of the non-degeneracy of $(\text{na})[\alpha^0]$; for the present we will confine attention to just one property: if $S \in \mathfrak{S}$, then

$$((\text{DOM})[(\text{na})[\alpha^0]] \cap S \neq \varnothing) \Leftrightarrow (S \subset [\text{DOM}][(\text{na})[\alpha^0]])$$

We note one very general proposition (see [24, 25] and [28, Sec. 6.11]).

Proposition 5. Suppose $C \in \mathfrak{S}$ and

$$(\alpha^0(\omega) \in \mathbb{K}\ \forall \omega \in C) \vee (\alpha^0(\omega) \in \mathcal{K}\ \forall \omega \in C)$$

Then the following conditions are equivalent:

(1) $(\text{DOM})[(\text{na})[\alpha^0]] \cap C = \varnothing$;
(2) $C \backslash (\text{DOM})[(\text{na})[\alpha^0]] \neq \varnothing$;
(3) $\exists k \in \mathcal{N}$: $(\text{DOM})[\mathbf{\Gamma}^k(\alpha^0)] \cap C = \varnothing$;
(4) $\exists k \in \mathcal{N}_0$: $C \backslash (\text{DOM})[\mathbf{\Gamma}^k(\alpha^0)] \neq \varnothing$.

Discussion. Proposition 5 does not require the sequence (3.1) to converge to $(\text{na})[\alpha^0]$; the assertion is local in nature – the analysis relates to a cell C of the partition (5.2) and the fragment $((\text{na})[\alpha^0] | C)$ of the multifunction $(\text{na})[\alpha^0]$ on that cell. If there is some $\omega \in C$ for which $(\text{na})[\alpha^0](\omega) = \varnothing$ (see Condition 2 in Proposition 5) then, by Condition 3, some $n \in \mathcal{N}$ exists such that $\mathbf{\Gamma}^n(\alpha^0)(\tilde{\omega}) = \varnothing\ \forall \tilde{\omega} \in C$. Thus, the direct version of the PIM (see condition (3.1)) enables us, provided α^0 has natural topological properties on the cells of (5.2), to establish the property $(\text{na})[\alpha^0] \notin Q^0$ (and hence also $Q^0 \neq \varnothing$; see Theorem 1) without recourse to a limit procedure; moreover, this property is established universally in the sense of the choice of a signal from the appropriate cell. The property $\mathbf{\Gamma}^n(\alpha^0)(\omega) = \varnothing$ itself is factorized on the basis of (5.2), which facilitates verification of the condition $Q^0 \neq \varnothing$ (one can consider a "cell" analogue of the condition $\mathbf{\Gamma}^n(\alpha^0)(\omega) = \varnothing$ as a "safe" version of verifying the unsolvability of the basic problem in the class of quasistrategies).

6. A SIMPLE EXAMPLE OF A CONTROL PROBLEM

We will now consider the interpretation of the general constructions of the foregoing sections for a simple control problem. Suppose we are given two systems

$$\dot{z} = u, \quad u \in P_* \tag{6.1}$$

$$\dot{e} = v, \quad v \in Q_* \tag{6.2}$$

where z, e, u and v are n-dimensional vectors, and P_* and Q_* are non-empty compact sets in the n-dimensional arithmetic space $\mathbb{R}_n$; assume that P_* is convex (for clarity, we will confine our attention to so-called simple motions). The motions of systems (6.1) and (6.2) are considered in an interval $[t_0, \vartheta_0]$, where $t_0 \in \mathbb{R}$, $\vartheta_0 \in \mathbb{R}$ and $t_0 < \vartheta_0$. System (6.1) is to be solved with initial data $z(t_0) = z_0 \in \mathbb{R}_n$. All that is known of the initial state of system (6.2) is that it is a point of a non-empty bounded set E_*, $E_* \subset \mathbb{R}_n$.

Player I controls system (6.1) using any function $U = u(\cdot)$ from the set $\mathcal{U}$ of all Borel mappings from $[t_0, \vartheta_0]$ into P_*. Every function $U \in \mathcal{U}$ generates (starting from a point z_0) a trajectory $\varphi_U^{(1)}$ over $[t_0, \vartheta_0]$, obtained by integrating U over the intervals $[t_0, t[$, $t \in [t_0, \vartheta_0]$. The bundle $\{\varphi_U^{(1)}: U \in \mathcal{U}\}$ will henceforth be denoted by Z, assuming (see Section 2) that $Y = \mathbb{R}_n$. Of course, Z is a non-empty compact set in the space $\mathbf{C}$ of all continuous n-vector-valued functions in $[t_0, \vartheta_0]$ with the uniform convergence topology $\mathcal{T}$.

System (6.2) is controlled by player II, who applies to that end any function $V = v(\cdot)$ from a certain non-empty set $\mathcal{V}$ of Borel mappings from $[t_0, \vartheta_0]$ into Q_*. In addition, he may choose any vector of initial data $e_0 \in E_*$, after which (starting from the point e_0) a trajectory $\varphi_V^{(2)}[e_0]$ is generated over $[t_0, \vartheta_0]$, obtained by integrating the function V over the intervals $[t_0, t[$, $t \in [t_0, \vartheta_0]$. The bundle of all such trajectories $\varphi_V^{(2)}[e_0]$, $V \in \mathcal{V}$, $e_0 \in E_*$ will henceforth be denoted by $\mathbb{E}$, on the assumption (see Section 2) that $E = \mathbb{R}_n$. Of course, $\mathbb{E}$ is a non-empty set in $\mathbf{C}$ which is bounded in the sup-norm of $\mathbf{C}$. Given all these assumptions, it is natural to assume from now on that $X = [t_0, \vartheta_0]$.

Let $\|\cdot\|$ be, say, the Euclidean norm in $\mathbb{R}_n$, $\delta \in]0, \infty[$. Putting $\Upsilon = \mathbb{R}_n$, we define the operator Λ of (2.1) as follows: if $e \in \mathbb{E}$, then $\Lambda(e)$ is defined to be the set of all functions from $[t_0, \vartheta_0]$ into $\mathbb{R}_n$, for each of which

$$\|e(t) - \omega(t)\| \leq \delta \ \forall t \in [t_0, \vartheta_0]$$

This conditions corresponds to the action of additive δ-bounded noise in the measurement channel. Following definition (2.2) with the above specific choice of Λ, we conclude that Ω is a non-empty set whose elements are functions from $[t_0, \vartheta_0]$ into $\mathbb{R}_n$. As a consequence, the operator (2.3) pairs each such vector-valued function with a non-empty subset of the bundle $\mathbb{E}$. In accordance with what was said in Section 2, we fix a set M, $M \subset Z \times \mathbb{E}$; suppose

$$M[e] \triangleq \{z \in Z \mid (z, e) \in M\} \ \forall e \in \mathbb{E} \tag{6.3}$$

Endowing the set $Y = \mathbb{R}_n$ with the usual coordinatewise topology, we obtain in the form ϑ the natural pointwise convergence topology on the set Z [31, Chap. 7]. In this connection it is useful to point out the relation with the uniform convergence topology $\mathbf{t}$ on Z induced (in Z) from $(\mathbf{C}, \mathcal{T})$ [31, p. 77]. Of course, $\mathbf{t}$ is the topology of Z generated by the uniform convergence metric; under those conditions [32, p. 174] $\vartheta \subset \mathbf{t}$, and therefore every subset of Z which is compact in the topological space $(Z, \mathbf{t})$ is also compact in (Z, ϑ), that is, it is an element of $\mathbb{K}$. Let us assume now that every set (6.3) is closed in the topological space $(Z, \mathbf{t})$.

In relation to the family $\mathcal{X}$ we define

$$\mathcal{X} \triangleq \{[t_0, t]: t \in [t_0, \vartheta_0]\}$$

Clearly, such a family $\mathcal{X}$ is the basis of a filter in X. Thus, the general formulation of the problem in Section 2 has been specialized; in the process, condition (2.5) becomes the usual requirement that the multivalued response α to signals developing in time is unpredictable. The meaning of α^0 is also clear: associated with every possible signal ω is a bundle of trajectories in Z implementing an element M paired with any trajectory in $\mathbb{E}$ that may be generated by the signal.

We will now consider the topological properties of α^0 and show that $\alpha^0(\omega)$, $\omega \in \Omega$, are compact subsets in $(Z, \mathbf{t})$ of Z. The space $(Z, \mathbf{t})$ itself is compact. Thus it will suffice to show that these sets, the values of α^0, are closed. Fix $\omega \in \Omega$. Consider the set $\alpha^0(\omega)$ (see definition (2.4)). Since the compact space $(Z, \mathbf{t})$ is metrizable, it will suffice to prove sequential closure. Let the function $z \in Z$ be such that some sequence $(z_i)_{i \in \mathcal{N}}$ in $\alpha^0(\omega)$ converges to z in $(Z, \mathbf{t})$. Choose an arbitrary $e \in \mathbb{L}(\omega)$. It follows from definition (2.4) that $(z_i, e) \in M \ \forall i \in \mathcal{N}$. In other words (see property (6.3)), $(z_i)_{i \in \mathcal{N}}$ is a sequence in $M[e]$ converging to z in $(Z, \mathbf{t})$. Since $M[e]$ is closed, we have $z \in M[e]$, which means that, as the choice of e was arbitrary, we have $z \in \alpha^0(\omega)$ (see properties (2.4) and (6.3)). Thus, the set $\alpha^0(\omega)$ is closed in $(Z, \mathbf{t})$, and so it is compact there as well. Taking the relation between ϑ and $\mathbf{t}$ into consideration, we conclude that the set $\alpha^0(\omega)$ is compact in (Z, ϑ). Since the choice of ω was arbitrary, this shows that $\alpha^0 \in \mathbb{K}^\Omega$. By Proposition 4, we have $(\mathrm{na})[\alpha^0] = \mathbf{\Gamma}^\infty(\alpha^0)$, and Theorem 3 yields the necessary and sufficient conditions that $Q^0 \neq \varnothing$. Namely, this condition holds if and only if $\mathbf{\Gamma}^\infty(\alpha^0)(\omega) \neq \varnothing \ \forall \omega \in \Omega$.

Of course, one might also use relations (4.3), that is, consider the intersection of the images of the set (4.5) under operators $\mathbf{\Gamma}_{[t_0, t]}$, $t \in [t_0, \vartheta_0]$ and try to find in that intersection a multifunction $\mathfrak{D}^0_{\mathbb{K}}$ with the property $\mathfrak{D}^0_{\mathbb{K}}(\omega) \neq \varnothing \ \forall \omega \in \Omega$.

I have had the good fortune to be able to discuss these results with Professor N. N. Krasovskii, beginning from his very earliest research relating to iterative constructions, which were presented at his seminar. These discussions were of great value in establishing and developing this area of research. I am also deeply indebted to Professor Krasovskii for his unfailing support.

This research was supported financially by the Russian Foundation for Basic Research (03-01-00415 and 04-01-96093).

REFERENCES

1. ISAACS, R., *Differential Games*. Wiley, New York, 1965.
2. KRASOVSKII, N. N., *Game-Theoretic Problems on the Meeting of Motions*. Nauka, Moscow, 1970.
3. KRASOVSKII, N. N. and SUBBOTIN, A. I., *Positional Differential Games*. Nauka, Moscow, 1974.
4. KRASOVSKII, N. N., *Control of a Dynamical System. The Problem of the Minimum of the Guaranteed Result*. Nauka, Moscow, 1985.
5. SUBBOTIN, A. I. and CHENTSOV, A. G., *Optimization of Guarantee in Control Problems*. Nauka, Moscow, 1981.
6. SUBBOTIN, A. I., *Minimax Inequalities and the Hamilton–Jacobi Equations*. Nauka, Moscow, 1991.
7. PONTRYAGIN, S., On the theory of differential games. *Uspekhi Mat. Nauk*, 1966, **21**, 4, 219–274.
8. PSHENICHNYI, B. N., The structure of differential games. *Dokl. Akad. Nauk SSSR*, 1969, **184**, 2, 285–287.
9. CHENTSOV, A. G., The structure of a game-theoretic problem of approach. *Dokl. Akad. Nauk SSSR*, 1975, **224**, 6, 1272–1275.
10. CHENTSOV, A. G., A game-theoretic problem of approach at a given instant of time. *Mat. Sbornik*, 1976, **99**, 3, 394–420.
11. CHENTSOV, A. G., A game-theoretic problem of aiming. *Dokl. Akad. Nauk SSSR*, 1976, **226**, 1, 73–76.
12. CHENTSOV, A. G., A game-theoretic problem of aiming with information memory. *Dokl. Akad. Nauk SSSR*, 1976, **227**, 2, 306–308.
13. KRASOVSKII, N. N. and CHENTSOV, A. G., On the design of differential games, I. *Probl. Control and Inform. Theory*, 1977, **6**, 5–6, 381–395.
14. KRASOVSKII, N. N. and CHENTSOV, A. G., On the design of differential games, II. *Probl. Control and Inform. Theory*, 1979, **8**, 1, 3–11.
15. ROXIN, E., Axiomatic approach in differential games. *J. Optim. Theory Appl.*, 1969, **3**, 153–163.
16. ELLIOT, R. J. and KALTON, N. J., *The Existence of Value in Differential Games* (Mem. Am. Math. Soc. 126). American Mathematical Society, Providence, RI, 1972.
17. CHISTYAKOV, S. V., The solution of game-theoretic problems of pursuit. *Prikl. Mat. Mekh.*, 1977, **41**, 5, 825–832.
18. MELIKYAN, A. A., The value of a game in a linear differential game of approach. *Dokl. Akad. Nauk SSSR*, 1977, **237**, 3, 521–524.
19. UKHOBOTOV, V. I., The construction of a stable bridge for a class of linear games. *Prikl. Mat. Mekh.*, 1977, **41**, 2, 358–361.
20. KURZHANSKII, A. B., *Control and Observation under Conditions of Uncertainty*. Nauka, Moscow, 1977.
21. OSIPOV, Yu. S., An informational game problem. In *Lecture Notes in Computer Science*, Vol. 27. Springer, Berlin, 1975, 482–486.
22. KRYAZHIMSKII, A. V., Alternative in a linear pursuit-evasion game with incomplete information. *Dokl. Akad. Nauk SSSR*, 1976, **230**, 4, 773–776.
23. CHENTSOV, A. G., The problem of the iterative realization of non-predictive multivalued mappings. *Izv. Vuzov. Mat.*, 2000, 3, 66–76.
24. CHENTSOV, A. G., Non-predictive multivalued mappings and their construction by the programmed iteration method, I. *Differents. Uravn.*, 2001, 4, 470–480.
25. CHENTSOV, A. G., Non-predictive multivalued mappings and their construction by the programmed iteration method, II. *Differents. Uravn.*, 2001, 5, 679–688.
26. KURATOWSKI, K. and MOSTOWSKI, A., *Set Theory*. North-Holland, Amsterdam, 1967.
27. NEVEU, J., *Bases Mathématiques du Calcul des Probabilités*. Masson et Cie., Paris, 1969.
28. CHENTSOV, A. G. and MORINA, S. I., *Extensions and Relaxations*. Kluwer, Dordrecht, 2003.
29. CHENTSOV, A. G., *The Method of Programmed Iterations for a Pursuit–Evasion Differential Game*. Sverdlovsk, 1979. Dep. at VINITI No. 1933–79 Dep.
30. CHENTSOV, A. G., MORINA, S. I. and ZOBNIN, B. B., On some constructions of control by systems with a varying structure. *Mathematics and Computers in Simulation*, 1999, **49**, 6, 319–334.
31. KELLEY, J. L., *General Topology*. Springer, New York, 1975.
32. ENGELKING, R., *General Topology*. PWN, Warsaw, 1977.

Translated by D.L.